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Stone–Weierstrass theorems revisited

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Abstract

We prove strengthened and unified forms of vector-valued versions of the Stone–Weierstrass theorem. This is possible by using an appropriate factorization of a topological space, instead of the traditional localizability. Our main Theorem 7 generalizes and unifies number of known results. Applications from the last section include new versions in the scalar case, as well as simultaneous approximation and interpolation under additional constraints.

1. Introduction and notations

Throughout this paper, T denotes a topological space, X a Hausdorff locally convex space over the scalar field $\Gamma \in \{\mathbb{R}, \mathbb{C}\}$, and $C(T, X)$ the linear space of all X-valued continuous functions on T. Many generalized Stone–Weierstrass theorems are intended to describe the closure of a subset $E \subset \mathcal{H}$, in vector subspaces $\mathcal{H} \subset C(T, X)$ endowed with various linear topologies. Typically, such results consider a nonempty subset $S \subset C(T, \Gamma)$, subject to one of the following conditions:

$$\varphi E + (1 - \varphi)E \subset E \quad \text{for every } \varphi \in S, \tag{1}$$

$$S \cdot E \subset E. \tag{2}$$

The generality of this approach also consists in the fact that one may take $S = E$ in the case of a subalgebra $E \subset C(T, \Gamma)$.

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Let us recall that every continuous $\varphi : T \to [0, 1]$ satisfying (1), is said to be a *multiplier* of *E*. Thus, a set of multipliers is by definition a subset of

$$C(T, [0, 1]) := \{\varphi : T \to [0, 1] \mid \varphi \text{ is continuous}\}.$$

A set *F* of functions defined on *T* (and taking values in various sets) is said to be *separating*, if for all distinct $t, s \in T$, we have $f(t) \neq f(s)$ for some $f \in F$. From the viewpoint of separating capabilities of *S*, there are two kinds of results.

(A) If *S* is required to be separating, then *T* needs to be Hausdorff. These assumptions lead to powerful results. Nevertheless, such theorems cannot be applied in the simple and very frequent case of a nonseparating subalgebra of $C(T, \Gamma)$, since no separating *S* can be found.

(B) If *S* is not required to be separating, the problem is reduced to a similar one on each *S*-equivalence class (which is a subset of *T*). Since all functions from *S* are constant on every such class, conditions as (1) and (2) are less useful, and there is no Stone–Weierstrass-type theorem applicable to the reduced problem.

Our purpose is to establish a good compromise between (A) and (B). We will show that the same conclusions can be obtained if we replace the assumptions from (A) by the weaker condition

$$\rho_S \subset \rho_E, \tag{3}$$

where ρ_S and ρ_E are the equivalence relations defined on *T* by *S* and *E* (see Section 2.1 for details). Roughly speaking, (3) says that *S* is "more separating" than *E*. Note that (3) holds whenever *S* is separating. Also, (2) and (3) hold if *E* is a subalgebra of $C(T, \Gamma)$ and $S = E$.

Let $\mathcal{V}_X(0)$ denote the set of all convex open neighborhoods of the origin in *X*. The following notations for vector subspaces of $C(T, X)$ are standard:

$$C_c(T, X) := \{u \in C(T, X) \mid \operatorname{supp} u := \overline{u^{-1}(X \setminus \{0\})} \text{ is compact}\},$$
$$C_0(T, X) := \{u \in C(T, X) \mid u^{-1}(X \setminus W) \text{ is compact } \forall\, W \in \mathcal{V}_X(0)\},$$
$$C_b(T, X) := \{u \in C(T, X) \mid u(T) \text{ is bounded}\}.$$

Here and elsewhere, *compact* means that every open covering has a finite subcovering (without requiring Hausdorff separation). It is obvious that

$$C_c(T, X) \subset C_0(T, X) \subset C_b(T, X) \subset C(T, X).$$

General setting:

From now on, $\mathcal{H}$ will mean any of the following three locally convex spaces (always equipped with the topology specified below):

(c) $C(T, X)$, with the topology τ_c of uniform convergence on all compact subsets of *T* (the compact-open topology),

(u) $C_0(T, X)$, with the topology τ_u of uniform convergence on *T*,

(β) $C_b(T, X)$, with the strict topology τ_β, if *T* is a locally compact Hausdorff space. Recall that the strict topology is defined by all weighted seminorms of the form

$$p_w : C_b(T, X) \to \mathbb{R}_+, \quad p_w(u) := \sup_{t \in T} p(w(t)u(t))$$

with $w \in C_0(T, \mathbb{R})$ and continuous seminorm $p : X \to \mathbb{R}_+$.

The closure in $\mathcal{H}$ of an arbitrary subset $E \subset \mathcal{H}$ will be denoted by $\overline{E}_{\mathcal{H}}$.

Note that on $C_0(T, X)$, we have $\tau_c \leqslant \tau_u$ (meaning that the second topology is stronger). If T is a locally compact Hausdorff space, we also have $\tau_c \leqslant \tau_\beta \leqslant \tau_u$ on $C_0(T, X)$. All three above cases coincide if T is a compact Hausdorff space. Also, in the scalar case $X = \Gamma$, the vector space $\mathcal{H}$ is a subalgebra of $C(T, \Gamma)$.

For applications it is useful to note that if $E \subset \mathcal{H}_0 \subset \mathcal{H}$, then

$$\overline{E}_{\mathcal{H}_0} = \overline{E}_{\mathcal{H}} \cap \mathcal{H}_0,$$

where $\mathcal{H}_0$ is considered as topological subspace (not necessarily linear) of $\mathcal{H}$. For instance, the τ_c-closure of E in $\mathcal{H}_0 \supset E$ can be described in this way.

2. Needed facts

2.1. Factorization of T

Let F be a set of functions defined on T, each $f \in F$ taking its values in a set Y_f. The equivalence relation ρ_F defined by F on T is

$$(t, s) \in \rho_F \iff f(t) = f(s) \quad \text{for every } f \in F.$$

The points $t, s \in T$ are said to be ρ_F*-distinct*, if $(t, s) \notin \rho_F$. For every $t \in T$, let t_F denote its ρ_F-class. The quotient set and the canonical surjection are

$$T_F := T/\rho_F = \{t_F \mid t \in T\} \text{ and } \pi_F : T \to T_F,\ \pi_F(t) = t_F.$$

A function $u : T \to Y$ factorizes as $u = \widehat{u} \circ \pi_F$ for some $\widehat{u} : T_F \to Y$, if and only if $\rho_F \subset \rho_{\{u\}} =: \rho_u$. In this case $\widehat{u}$ is unique, since π_F is a surjection. In particular, every $f \in F$ factorizes uniquely as

$$f = \widehat{f} \circ \pi_F, \quad \widehat{f} : T_F \to Y_f$$

and $\widehat{F} := \{\widehat{f} \mid f \in F\}$ obviously separates T_F. The quotient topology on T_F is

$$\{D \subset T_F \mid \pi_F^{-1}(D) \text{ is open in } T\}. \tag{4}$$

If Y is a topological space and $u : T \to Y$ factorizes as $u = \widehat{u} \circ \pi_F$, then u is continuous, if and only if $\widehat{u}$ is, by (4).

Proposition 1. *If every function $f \in F$ is continuous with respect to some Hausdorff topology on Y_f, then the quotient topological space T_F is Hausdorff.*

Proof. The proof is straightforward (every $\widehat{f}$ is continuous on the quotient space T_F, whose points are separated by $\widehat{F}$). □

Now assume (3) holds for E and S. Thus, taking in the above construction $F = S$ leads to the equivalence relation ρ_S, and consequently to

(a) the Hausdorff quotient space $T_S = T/\rho_S$ (even if T is not Hausdorff),
(b) the subsets $\widehat{E} = \{\widehat{v} \mid v \in E\} \subset C(T_S, X)$ and $\widehat{S} = \{\widehat{\varphi} \mid \varphi \in S\} \subset C(T_S, \Gamma)$.

Note that $\widehat{S}$ separates T_S, and that $\widehat{u} \in C_0(T_S, X)$ whenever $u \in C_0(T, X)$ (because $\widehat{u}^{-1}(G) = \pi_S(u^{-1}(G))$ for every $G \subset X$).

Remark 2. An important role in our study will be played by the set

$$\widetilde{E} := \{v \in C(T, X) \mid \rho_E \subset \rho_v,\ v(t) \in \overline{E(t)} \text{ for every } t \in T\}, \tag{5}$$

where $E(t) := \{v(t) \mid v \in E\}$ for every $t \in T$. Each function $v \in \widetilde{E}$ factorizes uniquely as $v = \widehat{v} \circ \pi_S$, for some $\widehat{v} \in C(T_S, X)$. We have the inclusion

$$\overline{E}_{\mathcal{H}} \subset \widetilde{E} \cap \mathcal{H}. \tag{6}$$

Indeed, if $u \in \overline{E}_{\mathcal{H}}$, then u belongs to the closure of E in $\mathcal{H}$ with respect to the pointwise convergence topology (weaker than the topology of $\mathcal{H}$). Hence, u is constant on each ρ_E-class, and so $u \in \widetilde{E}$.

The set $\widetilde{E}$ is important because Stone–Weierstrass theorems typically state that various hypotheses imply equality in (6).

Proposition 3. *Assume* (3) *holds together with one of conditions* (1),(2). *Let* $u : T \to X$, *such that* $u(t) \in \overline{E(t)}$ *for every* $t \in T$. *Then*

$$\rho_E \subset \rho_u \iff \rho_S \subset \rho_u.$$

Hence, in the definition (5) *of* $\widetilde{E}$ *we can replace* ρ_E *by* ρ_S. *If one of the sets* E, S *is separating and if* $E(t)$ *is dense in* X *for each* $t \in T$, *then* $\widetilde{E} = C(T, X)$.

Proof. Since (3) holds, we only need to prove "$\Leftarrow$". To show this, suppose that $\rho_S \subset \rho_u$, but there exists $(t, s) \in \rho_E \setminus \rho_u \subset \rho_E \setminus \rho_S$. Hence, $\varphi(t) \neq \varphi(s)$ for some $\varphi \in S$. Fix $v, w \in E$, and set $x := v(t) = v(s),\ y := w(t) = w(s)$. We claim that $x = y$. We need to analyze two cases.

Case 1: If (1) holds for S and E, then $(1 - \varphi)v + \varphi w \in E$ leads by $(t, s) \in \rho_E$ to $(1 - \varphi(t))x + \varphi(t)y = (1 - \varphi(s))x + \varphi(s)y$, which yields $x = y$.

Case 2: If (2) holds, then $\varphi v \in E$ leads by $(t, s) \in \rho_E$ to $\varphi(t)x = \varphi(s)x$, which yields $x = 0$. Similarly, $\varphi w \in E$ forces $y = 0$, and consequently, $x = y$.

We conclude that $x = y$. Since v and w were arbitrarily fixed, it follows that $E(t) = E(s) = \{x\}$. We thus get $u(t) = x = u(s)$, which contradicts $(t, s) \notin \rho_u$. Hence, we must have $\rho_E \subset \rho_u$. The last part is immediate. $\square$

The last part of the above proposition may be used in order to convert results describing the closure $\overline{E}_{\mathcal{H}}$ into density results. Indeed, if equality holds in (6) and if $\widetilde{E} = C(T, X)$, then E is dense in $\mathcal{H}$. The following remark is useful.

Remark 4. Let $u \in C(T, X)$ be fixed. If for all points $t, s \in T$ and each neighborhood $W \in \mathcal{V}_X(0)$ there exists $v \in E$ such that

$$v(t) - u(t) \in W \quad \text{and} \quad v(s) - u(s) \in W,$$

then $u \in \widetilde{E}$.

A first application of the factorization described in this section: it leads at once from Theorem 2 of Jewett [6] to Theorem 2 of Prolla [7]. Our discussion from Section 4.4 will show that factorization is by itself an efficient tool for the study of density problems in functions spaces.

For various considerations on topological spaces with equivalence relations, we refer the reader to Dugundji [5, pp. 125–130].

2.2. *Self-adjoint sets*

In Section 4, we will describe the closure of a vector subspace $E \subset \mathcal{H}$, in the presence of a set S satisfying (2). In the complex case ($\Gamma = \mathbb{C}$) it is natural to impose self-adjointness conditions on S or E.

Definition 5. (i) By *involution* on X we will mean any $\mathbb{R}$-linear operator $X \ni x \mapsto x^\star \in X$, such that $(x^\star)^\star = x$ and $(\lambda x)^\star = \lambda^\star x^\star$ for all $x \in X$ and $\lambda \in \Gamma$, where $\lambda^\star$ stands for the complex conjugate of the number λ. Any continuous involution on X induces on $C(T, X)$ an associated involution

$$v \mapsto v^\star, \quad v^\star(t) := (v(t))^\star \text{ for all } v \in C(T, X), t \in T. \tag{7}$$

On the field Γ we always consider the complex conjugation as involution.
(ii) The set $E \subset C(T, X)$ is called *self-adjoint* with respect to the continuous involution $x \mapsto x^\star$ on X, if and only if E is invariant for the associated involution (7), that is,

$$\{v^\star \mid v \in E\} = E.$$

Self-adjointness of $S \subset C(T, \Gamma)$ is defined in the same way. [1]

Let us note that any $S \subset C(T, \mathbb{R})$ is automatically self-adjoint. Also, the identity operator of any real vector space is an involution. Because of these facts, we will be able to state our results without distinction between the complex and the real case, since in the latter self-adjointness produces no restriction.

2.3. *A known approximation lemma*

The following lemma is taken from Prolla [8, Lemma 3, p. 302]; see also Prolla [10, Lemma 2.2, p. 174].

Lemma 6. *Assume T is a compact Hausdorff space. Let $M \subset C(T, [0, 1])$ be a separating subset satisfying the property of von Neumann*

$$1 - \varphi \in M \text{ and } \varphi\psi \in M \quad \text{for all } \varphi, \psi \in M. \tag{8}$$

[1] With respect to the complex conjugation on $\Gamma \subset \mathbb{C}$.

Let $s \in V \subset T$, with V open. Then V contains an open neighborhood U of s, such that for every $\delta \in]0, \frac{1}{2}[$, there exists a function $\varphi \in M$, with

$$\begin{cases} \varphi(t) > 1 - \delta \text{ for every } t \in U, \\ \varphi(t) < \delta \text{ for every } t \in T \setminus V. \end{cases}$$

The above result in the case of $M = C(T, [0,1])$ was first proved by Brosowski and Deutsch [2, Lemma 1, p. 90].

3. Stone–Weierstrass theorems for subsets

Our following result generalizes the main Theorem 1 from Prolla [8] in three ways:

1. T is arbitrary (not necessarily compact or Hausdorff),
2. X is locally convex (not necessarily normable),
3. S need not be separating.

This makes it able to subsume various known results; see for instance Prolla [9, Theorem 1.11, p. 13, Corollary 6.3, p. 118, Corollary 7.3, p. 127] and Timofte [12, Corollary 1 and Theorem 2, p. 294]. Also, our theorem *deals with six cases*: two for the scalar field Γ, combined with three for the locally convex space $\mathcal{H}$.

Theorem 7. *Assume* (3) *holds for some set of multipliers of $E \subset \mathcal{H}$. Then*

$$\overline{E}_{\mathcal{H}} = \widetilde{E} \cap \mathcal{H}.$$

In particular, if E is separating and $E(t)$ is dense in X for every $t \in T$, then the subset E is dense in $\mathcal{H}$.

Proof. By (6), we only need to prove in each case that $\overline{E}_{\mathcal{H}} \supset \widetilde{E} \cap \mathcal{H}$. Throughout the proofs we shall write the closure of a set of functions by using a lower index specifying the space in which the closure is considered, as well as an upper index (u,c, or β) for the topology of this space.

Case (u): $\mathcal{H} = C_0(T, X)$. We need to prove that $\widetilde{E} \cap C_0(T, X) \subset \overline{E}^{\mathrm{u}}_{C_0(T,X)}$. Fix $u \in \widetilde{E} \cap C_0(T, X)$ and $W \in \mathcal{V}_X(0)$. In order to show that $(v - u)(T) \subset W$ for some $v \in E$, we shall analyze two subcases.

Subcase (u1): If S is separating (and hence T is Hausdorff), let $M \subset C(T, [0,1])$ denote the set of all multipliers of E. We have $S \subset M$, and M satisfies (8) (see Prolla [8, p. 301]). Let $W_0 := \frac{1}{3}W \in \mathcal{V}_X(0)$. The set $K := u^{-1}(X \setminus W_0)$ is compact, since $u \in C_0(T, X)$. For each $s \in T$, choose

$$v_s \in E, \text{ such that } (v_s - u)(s) \in W_0,$$
$$K_s := K \cup v_s^{-1}(X \setminus W_0), \quad G_s := (v_s - u)^{-1}(W_0).$$

Hence, K_s is compact and G_s is open in T. Select a point $s_1 \in T$ arbitrarily. For every $s \in K_{s_1} \setminus G_{s_1}$, set

$$T_s := K_s \cup K_{s_1}, \quad V_s := G_s \cap T_s.$$

We have $s \in V_s \subset T_s$. Since T_s is compact, V_s is open in T_s, and the separating set $M|_{T_s} \subset C(T_s, [0,1])$ satisfies (8), we can choose a neighborhood U_s of s in T_S, with $U_s \subset V_s$, and with the property from Lemma 6. We have $U_s = D_s \cap T_s$ for some open $D_s \subset G_s$. As $K_{s_1} \setminus G_{s_1}$ is compact, we have $K_{s_1} \setminus G_{s_1} \subset \bigcup_{j=2}^n D_{s_j}$ for some finite set $\{s_2, \ldots, s_n\} \subset K_{s_1} \setminus G_{s_1}$. For simplicity of notation, we will write $v_j, K_j, G_j, T_j, V_j, U_j, D_j$ instead of $v_{s_j}, K_{s_j}, G_{s_j}, T_{s_j}, V_{s_j}, U_{s_j}, D_{s_j}$, for all j. Since the set $B := \bigcup_{j=1}^n (v_j - u)(T)$ is bounded in X, we have $B \subset \mu W_0$ for some $\mu > 0$. Choose $\delta \in]0, \frac{1}{2}[$, such that $\delta n \mu < 1$. According to Lemma 6, for each $j \in \{2, \ldots, n-1\}$, there exists $\varphi_j \in M$, such that

$$\varphi_j(t) > 1 - \delta \quad \text{for every } t \in U_j, \tag{9}$$

$$\varphi_j(t) < \delta \quad \text{for every } t \in T_j \setminus V_j, \tag{10}$$

Since M satisfies (8), we can define $\psi_1, \ldots, \psi_n \in M$,

$$\begin{aligned}
&\psi_2 := \varphi_2,\\
&\psi_3 := (1-\varphi_2)\varphi_3,\\
&\cdots\\
&\psi_n := (1-\varphi_2)(1-\varphi_3)\cdots(1-\varphi_{n-1})\varphi_n,\\
&\psi_1 := (1-\varphi_2)(1-\varphi_3)\cdots(1-\varphi_{n-1})(1-\varphi_n).
\end{aligned}$$

It is easily seen that $\sum_{j=1}^n \psi_j = 1$. Let us define $v := \sum_{j=1}^n \psi_j v_j \in C_0(T, X)$. In order to prove that $v \in E$, consider

$$w_0 = v_1, \; w_{i+1} = \varphi_{n-i} v_{n-i} + (1 - \varphi_{n-i}) w_i \quad \text{for every } i \in \{0, \ldots, n-2\}.$$

An easy induction shows that

$$w_i \in E, \quad v = \sum_{j=2}^{n-i} \psi_j v_j + \prod_{j=2}^{n-i} (1 - \varphi_j) w_i \quad \text{for every } i \in \{0, \ldots, n-2\}.$$

We thus get $v = \psi_2 v_2 + (1-\varphi_2) w_{n-2} = \varphi_2 v_2 + (1 - \varphi_2) w_{n-2} \in E$. Let us finally prove that $(v-u)(T) \subset W$. Fix $t \in T$. We obviously have $(v-u)(t) = \sum_{j=1}^n \psi_j(t)(v_j - u)(t)$ and $\{1, \ldots, n\} = J_1 \cup J_2 \cup J_3$, where

$$J_1 := \{j \mid t \in V_j\}, \quad J_2 := \{j \mid t \in T_j \setminus V_j\}, \quad J_3 := \{j \mid t \in T \setminus T_j\}.$$

For every $j \in J_1$ we have $t \in V_j \subset G_j$, and so $(v_j - u)(t) \in W_0$. For $j \in J_3$ we have $t \notin K_j$, and so $u(t), v_j(t) \in W_0$, which yields $(v_j - u)(t) \in 2W_0$. Thus,

$$\sum_{j \in J_1 \cup J_3} \psi_j(t)(v_j - u)(t) \in \sum_{j \in J_1} \psi_j(t) W_0 + 2 \sum_{j \in J_3} \psi_j(t) W_0 \subset 2W_0. \tag{11}$$

Let $j \in J_2$. We claim that $\psi_j(t) < \delta$. Indeed, if $j \geqslant 2$, then $t \in T_j \setminus V_j$ forces $\psi_j(t) \leqslant \varphi_j(t) < \delta$, by (10). If $j = 1$, then $t \in T_1 \setminus V_1 = K_1 \setminus G_1 \subset \bigcup_{i=2}^n D_i$, and so $t \in D_i$ for some $i \geqslant 2$. This yields $t \in D_i \cap K_1 \subset D_i \cap T_i = U_i$, which leads by (9) to $\psi_1(t) \leqslant 1 - \varphi_i(t) < \delta$. It follows that

$$\sum_{j \in J_2} \psi_j(t)(v_j - u)(t) \in \sum_{j \in J_2} \psi_j(t) B \subset \mu \sum_{j \in J_2} \psi_j(t) W_0 \subset \delta n \mu W_0 \subset W_0. \tag{12}$$

According to (11) and (12), we have $(v-u)(t) \in 2W_0 + W_0 = 3W_0 = W$. We conclude that $(v-u)(T) \subset W$.

Subcase (u2): If S is not separating, since $\rho_S \subset \rho_E \subset \rho_u$, we can consider

$$T_S = T/\rho_s, \quad \widehat{S} \subset C(T_S, [0,1]), \quad \widehat{E} \subset C_0(T_S, X), \quad \widehat{u} \in C_0(T_S, X).$$

But $\widehat{S}$ is a separating set of multipliers of $\widehat{E}$, and $\widehat{u}(t_S) = u(t) \in \overline{E(t)} = \overline{\widehat{E}(t_S)}$ for every $t \in T$, that is, $\widehat{u} \in \widetilde{\widehat{E}}$. By the conclusion of subcase (u1), it follows that $(v-u)(T) = (\widehat{v} - \widehat{u})(T_S) \subset W$ for some $v \in E$.

Case (c): $\mathcal{H} = C(T,X)$. We need to prove that $\widetilde{E} \subset \overline{E}^{\mathrm{c}}_{C(T,X)}$. Fix $u \in \widetilde{E}$, and a compact $K \subset T$. Since $E|_K \subset C(K,X) = C_0(K,X)$ and $\rho_{S|_K} \subset \rho_{E|_K}$, by the already proved case (u) we get $\overline{(E|_K)}^{\mathrm{u}}_{C_0(K,X)} = \widetilde{(E|_K)} \cap C_0(K,X) \ni u|_K$, and the conclusion follows.

Case (β): $\mathcal{H} = C_{\mathrm{b}}(T,X)$. We need to prove that $\widetilde{E} \cap C_{\mathrm{b}}(T,X) \subset \overline{E}^{\beta}_{C_{\mathrm{b}}(T,X)}$. Fix $u \in \widetilde{E} \cap C_{\mathrm{b}}(T,X)$, and $\omega \in C_0(T,\Gamma)$. Since $\omega E \subset C_0(T,X)$ and $\rho_S \subset \rho_{\omega E}$, by the already proved case (u) we get $\overline{(\omega E)}^{\mathrm{u}}_{C_0(T,X)} = \widetilde{(\omega E)} \cap C_0(T,X) \ni \omega u$, and the conclusion follows. $\square$

4. Applications

4.1. Stone–Weierstrass theorem for vector subspaces

In this section, we assume E to be a vector subspace of $\mathcal{H}$. In this case, conditions (1) and (2) are equivalent for any subset $S \subset C(T,\Gamma)$.

The following corollary generalizes Theorem 3 from Prolla [8] in the same three ways indicated in the previous section, as well as by considering a set S of scalar functions which are not necessarily real-valued (see also Remark 9).

Corollary 8. *Assume* (2) *and* (3) *hold for some self-adjoint* $S \subset C(T,\Gamma)$. *If* $\mathcal{H} \neq C(T,X)$, *also assume that* $S \subset C_{\mathrm{b}}(T,\Gamma)$. *Then*

$$\overline{E}_{\mathcal{H}} = \widetilde{E} \cap \mathcal{H}.$$

In particular, if E is separating and $E(t)$ is dense in X for every $t \in T$, then the subspace E is dense in $\mathcal{H}$.

Proof. *Cases* (u) and (β): $\mathcal{H} = C_0(T,X)$ or $\mathcal{H} = C_{\mathrm{b}}(T,X)$. Let us first observe that $S \subset C_{\mathrm{b}}(T,\Gamma)$. Therefore, we can consider

$$S_0 := \bigcup_{\varphi \in S} \left\{ \frac{\varphi^\star + \varphi + 2\|\varphi\|_\infty}{1 + 4\|\varphi\|_\infty}, \; \frac{\mathbf{i}(\varphi^\star - \varphi) + 2\|\varphi\|_\infty}{1 + 4\|\varphi\|_\infty} \right\},$$

where $\| \, \|_\infty$ stands for the supremum norm (and $\varphi^\star$ for the complex conjugate of φ). We see that $S_0 \subset C(T,[0,1])$ is a set of multipliers of E, and that $\rho_{S_0} = \rho_S \subset \rho_E$. Thus, the conclusion follows by Theorem 7.

Case (c): $\mathcal{H} = C(T, X)$. By (6), we only need to prove that $\overline{E}^{c}_{C(T,X)} \supset \widetilde{E}$. The proof is similar to that for the corresponding case of Theorem 7, and uses the conclusion of the above case (u).

Remark 9. If E is self-adjoint with respect to a continuous involution on X, then S need not be self-adjoint in Corollary 8.

Proof. Set $S^\star := \{\varphi^\star | \, \varphi \in S\}$. If $S \cdot E \subset E$ and $E^\star = E$ with respect to the involution associated to that on X, then

$$S^\star \cdot E = S^\star \cdot E^\star = (S \cdot E)^\star \subset E^\star = E,$$

and so $(S + S^\star) \cdot E \subset E$. Therefore, we can replace S by $S + S^\star \supset S$, since $\rho_{S+S^\star} \subset \rho_S$, and $S^\star \subset C_{\mathrm{b}}(T, \Gamma)$ if $S \subset C_{\mathrm{b}}(T, \Gamma)$. □

4.2. The scalar case $(X = \Gamma)$

In this section we assume that both E and $\mathcal{H} \supset E$ are vector subspaces of $C(T, \Gamma)$. According to our general setting from Section 1, $\mathcal{H}$ is a subalgebra of $C(T, \Gamma)$. Let us note that for every $t \in T$, we have $E(t) = \Gamma$ or $E(t) = \{0\}$, and so

$$\widetilde{E} = \{v \in C(T, \Gamma) \,|\, \rho_E \subset \rho_v, \; v(t) = 0 \text{ whenever } E(t) = \{0\}\}.$$

Definition 10. We define the *critical set* $\Lambda(E)$ of E to be the set of all scalars $\lambda \in \Gamma$ with $|\lambda| = 1$, such that there exist ρ_E-distinct points $t, s \in T$ satisfying

$$v(s) = \lambda v(t) \;\text{ for every } v \in E.$$

Remark 11. We have $1 \notin \Lambda(E)$. If $\Gamma = \mathbb{R}$, then $\Lambda(E) \subset \{-1\}$.

In the following theorem, the critical set $\Lambda(E)$ is subject to a fairly general condition, as Remark 13 will show.

Theorem 12. *Let the nonempty set of functions* $G \subset C(\Gamma^n, \Gamma) \setminus \Gamma$ *satisfy*

$$g(v_1, \ldots, v_n) \cdot w \in E \text{ for all } g \in G \text{ and } v_1, \ldots, v_n, w \in E. \tag{13}$$

Assume that for each $\lambda \in \Lambda(E)$, *we have* $g(\lambda x) \neq g(x)$ *for some* $g \in G$, $x \in \Gamma^n$, *and that one of the sets* G, E *is self-adjoint. Then*

$$\overline{E}_{\mathcal{H}} = \widetilde{E} \cap \mathcal{H}.$$

In particular, if E *is separating and* $E(t) \neq \{0\}$ *for every* $t \in T$, *then the subspace* E *is dense in* $\mathcal{H}$.

Proof. In (13), we used the notation $g(v_1, \ldots, v_n)$ for the map

$$T \ni t \mapsto g(v_1(t), \ldots, v_n(t)) \in \Gamma.$$

Let us first observe that we can assume that $n \geqslant 2$. Indeed, if $n = 1$ we may replace the set G by $\widetilde{G} = \{\widetilde{g} \mid g \in G\} \subset C(\Gamma^2, \Gamma) \setminus \Gamma$, where

$$\widetilde{g}(x_1, x_2) = g(x_1) \quad \text{for all } g \in G \text{ and } x_1, x_2 \in \Gamma.$$

Set $S := \{g(v_1, \ldots, v_n) \mid g \in G,\ v_1, \ldots, v_n \in E\} \subset C(T, \Gamma)$. We have $S \cdot E \subset E$ by (13). If $E \subset C_b(T, \Gamma)$, then $S \subset C_b(T, \Gamma)$, since $\overline{v(T)}$ is compact in Γ^n for every $v = (v_1, \ldots, v_n) \in E^n \subset C_b(T, \Gamma^n)$. The conclusion will follow by Corollary 8 and Remark 9, if we prove that $\rho_S \subset \rho_E$. Suppose that there exists $(t, s) \in \rho_S \setminus \rho_E$. Hence, $w(t) \neq w(s)$ for some $w \in E$, and we can assume that $w(t) = 1$. We now need to consider two cases.

Case 1: If for some $v = (v_1, \ldots, v_n) \in E^n$, the vectors $v(t), v(s) \in \Gamma^n$ are linearly independent, let us fix $g \in G$. For arbitrary $x \in \Gamma^n$, there exists a linear map $A : \Gamma^n \to \Gamma^n$, such that $A(v(t)) = x$ and $A(v(s)) = 0$. As $A \circ v \in E^n$ yields $\varphi := g(A \circ v) \in S$, we have $g(x) = \varphi(t) = \varphi(s) = g(0)$, since $(t, s) \in \rho_S$. We thus get $g \equiv g(0) \in \Gamma$, a contradiction.

Case 2: If (t, s) is not as in the first case, then there exists $\lambda \in \Gamma$, such that $v(s) = \lambda v(t)$ for every $v \in E$, because $n \geqslant 2$. Hence, $w(s) = \lambda w(t) = \lambda$. We claim that $|\lambda| = 1$, and that

$$g(\lambda x) = g(x) \quad \text{for all } g \in G,\ x \in \Gamma^n. \tag{14}$$

Let us fix $g \in G$ and $x \in \Gamma^n$. As $w \otimes x \in E^n$ yields $\varphi := g(w \otimes x) \in S$, we have $g(x) = \varphi(t) = \varphi(s) = g(\lambda x)$, since $(t, s) \in \rho_S$. Hence, (14) holds. Now suppose that $|\lambda| \neq 1$. We can assume that $|\lambda| < 1$, since otherwise we can use (14) with $1/\lambda$ instead of λ. As $g(x) = g(\lambda^k x)$ for every $k \in \mathbb{N}$, we have $g(x) = \lim_{k \to \infty} g(\lambda^k x) = g(0)$. Since $g \equiv g(0) \in \Gamma$ is a contradiction, we conclude that $|\lambda| = 1$. It follows that $\lambda \in \Lambda(E)$. According to the hypothesis, we have $h(\lambda y) \neq h(y)$ for some $h \in G$, $y \in \Gamma^n$, which contradicts (14). □

As both cases lead to contradictions, we conclude that $\rho_S \subset \rho_E$. The conclusion now follows by Corollary 8 and Remark 9. □

Remark 13. The hypothesis on $\Lambda(E)$ from Theorem 12 is satisfied in each of the following cases:

(a) $\Lambda(E) = \emptyset$,
(b) E is a subalgebra of $C(T, \Gamma)$,
(c) $1 \in E$,
(d) For all ρ_E-distinct $t, s \in T$, there exist functions $v_1, v_2 \in E$, such that

$$\begin{vmatrix} v_1(t) & v_1(s) \\ v_2(t) & v_2(s) \end{vmatrix} \neq 0,$$

(e) $\rho_{|E|} \subset \rho_E$ (where $|E| := \{|v| \mid v \in E\} \subset C(T, \mathbb{R})$),
(f) $G|_V$ is separating for some neighborhood $V \in \mathcal{V}_{\Gamma^n}(0)$,
(g) $\Gamma = \mathbb{R}$, and for all ρ_E-distinct points $t, s \in T$, we have $v(t) + v(s) \neq 0$ for some function $v \in E$,
(h) $\Gamma = \mathbb{R}$, and the functions from G are not all even.

Proof. The proof is straightforward. □

Since the case of a subalgebra is particularly important, we next state the corresponding corollary.

Corollary 14. *Assume E is a self-adjoint subalgebra of $\mathcal{H}$. Then*

$$\overline{E}_{\mathcal{H}} = \widetilde{E} \cap \mathcal{H}.$$

In particular, if E is separating and $E(t) \neq \{0\}$ for every $t \in T$, then the subalgebra E is dense in $\mathcal{H}$.

Theorem 12 and Remark 13 generate various Stone–Weierstrass-type results for vector subspaces $E \subset C(T, \Gamma)$. Some of them are quite strange, and far from being trivial. For instance, we can take $\Gamma = \mathbb{R}$, $n = 1$, and $G = \{g\}$ for any of the following functions $g \in C(\mathbb{R}, \mathbb{R})$:

$$g(x) = x|x|, \quad g(x) = x^3, \quad g(x) = \mathrm{e}^x, \quad g(x) = \sin x, \quad g(x) = \mathrm{e}^x \cos x.$$

We also have results of the following type:

Corollary 15. *If the subspace $E \subset \mathcal{H}$ is self-adjoint and satisfies*

$$\underbrace{E \cdot E \cdot \cdots \cdot E}_{2n \text{ factors}} \subset E, \tag{15}$$

for some fixed $n \geqslant 1$, then the conclusions of Theorem 12 *hold*.

Proof. For $g : \Gamma \to \Gamma$, $g(x) = x|x|^{2n-2}$, the set $G := \{g\} \subset C(\Gamma, \Gamma)$ is clearly separating, and E satisfies (13). Indeed, for all $v, w \in E$, we have

$$(g \circ v)w = |v|^{2n-2} vw = \left\{ \begin{array}{ll} v^{2n-1} w & \text{if } \Gamma = \mathbb{R} \\ (v^{\star})^{n-1} v^n w & \text{if } \Gamma = \mathbb{C} \end{array} \right] \in E \cdot E \cdot \cdots \cdot E \subset E,$$

and so (13) holds. Therefore, Theorem 12 can be applied. □

The above corollary is false for the product of $2n + 1$ factors E in (15). To see this, take $E \subset C([-1, 1], \mathbb{R})$ consisting of all odd continuous functions, and $\mathcal{H} = C([-1, 1], \mathbb{R})$. Then E is closed, but $E \neq \widetilde{E} = \{v \in \mathcal{H} \mid v(0) = 0\}$.

4.3. Approximations with constraints

In Timofte [12,13], special uniform approximations were considered (we should have called them "*support-range approximations*"). It was pointed out in [12] that uniform approximations satisfying relation (21) below have interesting applications:

- an equivalence between the possibility of vector-valued extension and that of uniform approximation, and some Tietze–Dugundji-type extension theorems (see Dugundji [5, p. 188] for the classical one),
- a short new proof of the Schauder–Tihonov fixed point theorem.

These were possible because of the particularly good control on approximant's range provided by (17) and (21). This motivates our next two results combining Theorems 2 and 3 from [12] with interpolation.

Theorem 16. *Consider a function $u \in C_0(T, X)$, a finite subset $F \subset T$, and a neighborhood $W \in \mathcal{V}_X(0)$. Then there exists a function $v \in C_c(T, \Gamma) \otimes X$ (algebraical tensor product), such that $v|_F = u|_F$, $\rho_u \subset \rho_v$, and*

$$(v-u)(T) \subset W, \quad \operatorname{supp} v \subset u^{-1}(X \setminus \{0\}), \tag{16}$$

$$v(T) \subset \operatorname{co}(u(T) \cup \{0\}). \tag{17}$$

Proof. The proof will be divided into 5 steps.

Step 1 (Reduction by factorization): Consider the quotient topological space $T_u := T/\rho_u$ (which is Hausdorff), the canonical surjection $\pi_u : T \to T_u$, the finite set $F_u := \pi_u(F)$, and the function $\omega \in C_0(T_u, X)$, such that $u = \omega \circ \pi_u$ (that is, $\omega = \widehat{u}$). We claim that $L := \omega^{-1}(X \setminus \{0\})$ is locally compact. To prove this, fix $\theta \in L$. Since $\omega(\theta) \neq 0$, choose $V \in \mathcal{V}_X(0)$, such that $\omega(\theta) \notin \overline{V}$. We clearly have $\theta \in \omega^{-1}(X \setminus \overline{V}) \subset \omega^{-1}(X \setminus V) \subset L$. As $\omega^{-1}(X \setminus \overline{V})$ is open in T_u, the set $\omega^{-1}(X \setminus V)$ is a compact neighborhood of θ in L. We conclude that L is locally compact.

Step 2 (Construction of the appropriate E and S): Let $E \subset C_0(T_u, X)$ denote the set consisting of all functions $\alpha \in C_c(T_u, \Gamma) \otimes X$ satisfying

$$\alpha|_{F_u} = \omega|_{F_u}, \quad \alpha(T_u) \subset \operatorname{co}(\omega(T_u) \cup \{0\}), \quad \operatorname{supp} \alpha \subset L. \tag{18}$$

It is easily seen that $S := C(T_u, [0,1])$ is a set of multipliers of E. We shall have established the theorem if we prove that

$$\omega \in \overline{E}^{\mathrm{u}}_{C_0(T_u, X)}. \tag{19}$$

Indeed, assume (19) holds, and choose $\alpha \in E$, such that $(\alpha - \omega)(T_u) \subset W$. Let us define $v := \alpha \circ \pi_u \in C_b(T, \Gamma) \otimes X$. By (18), we deduce that

$$v|_F = u|_F, \quad \rho_u = \rho_{\pi_u} \subset \rho_v, \quad (v-u)(T) = (\alpha - \omega)(T_u) \subset W,$$
$$v(T) = \alpha(T_u) \subset \operatorname{co}(\omega(T_u) \cup \{0\}) = \operatorname{co}(u(T) \cup \{0\}).$$

We claim that $v \in C_c(T, \Gamma) \otimes X$. As $K := \operatorname{supp} \alpha$ is compact and $0 \notin \omega(K)$, choose $V \in \mathcal{V}_X(0)$, such that $\overline{V} \cap \omega(K) = \emptyset$. Thus, $K \subset \omega^{-1}(X \setminus \overline{V}) \subset L$. Since K is compact and $\omega^{-1}(X \setminus \overline{V})$ is open in the locally compact L, there exists $\psi \in C(L, [0,1])$, such that $\psi|_K \equiv 1$ and $\operatorname{supp} \psi \subset \omega^{-1}(X \setminus \overline{V})$. Hence, $\varphi : T_u \to [0,1]$ defined by $\varphi|_L = \psi$, $\varphi|_{T_u \setminus L} \equiv 0$ is continuous, $\varphi|_K \equiv 1$, and $\operatorname{supp} \varphi \subset \omega^{-1}(X \setminus \overline{V})$. Therefore, $\alpha = \varphi\alpha$ yields $v = (\varphi \circ \pi_u)v$, which leads to

$$\operatorname{supp} v \subset \operatorname{supp}(\varphi \circ \pi_u) \subset \pi_u^{-1}(\operatorname{supp} \varphi) \subset \pi_u^{-1}(\omega^{-1}(X \setminus \overline{V})) \subset u^{-1}(X \setminus V).$$

As $u^{-1}(X \setminus V)$ is compact, we have $v = (\varphi \circ \pi_u)v \in C_c(T, \Gamma) \otimes X$, and so v satisfies all required properties. We are thus reduced to proving (19).

Step 3: We show that for every finite subset $A \subset T_u$, there exists $\alpha \in E$, such that $\alpha|_A = \omega|_A$. To prove this, fix such $A \subset T_u$, and set $M := (F_u \cup A) \cap L$. Thus, M is a finite subset of the locally compact L, which is open in T_u. Therefore, we can find in L pairwise disjoint compact neighborhoods $(U_\theta)_{\theta\in M}$ of the points $\theta \in M$. For each $\theta \in M$, there exists $\varphi_\theta \in C(T_u, [0,1])$, with $\varphi_\theta(\theta) = 1$ and the support supp φ_θ contained in the interior of U_θ (and hence, contained in L). Now define the function

$$\alpha := \sum_{\theta\in M} \varphi_\theta \cdot \omega(\theta) \in C_{\mathrm{c}}(T_u, \Gamma) \otimes X.$$

Clearly, $\alpha(T_u) \subset \mathrm{co}(\omega(M) \cup \{0\})$ and $\operatorname{supp} \alpha \subset \bigcup_{\theta\in M} \operatorname{supp} \varphi_\theta \subset L$. We also have $\alpha|_{F_u \cup A} = \omega|_{F_u \cup A}$, since $\alpha|_M = \omega|_M$ and $\alpha|_{T_u \setminus L} = \omega|_{T_u \setminus L} \equiv 0$. We conclude that $\alpha \in E$ and $\alpha|_A = \omega|_A$. By Remark 4, it follows that $\omega \in \widetilde{E}$.

Step 4: We prove that $\rho_S \subset \rho_E$. Let X^* denote the dual of X (that is, the set of all linear $f \in C(X, \Gamma)$), and set

$$S_0 := \{f \circ \alpha \mid f \in X^*, \alpha \in E\} \subset C_{\mathrm{b}}(T_u, \Gamma).$$

As X is Hausdorff, we have $\rho_E = \rho_{S_0} \supset \rho_{C_{\mathrm{b}}(T_u,\Gamma)} = \rho_S$. The last equality follows from $C_{\mathrm{b}}(T_u, \Gamma) = \{av + bw + c \mid v, w \in C(T_u, [0,1]),\ a, b, c \in \Gamma\}$.

Step 5 (Conclusion): Applying Theorem 7 finally shows that (19) holds. □

Corollary 17. *Assume T is compact. Consider a function $u \in C(T, X)$, a finite subset $F \subset T$, and a neighborhood $W \in \mathcal{V}_X(0)$. Then there exists a function $v \in C(T, \Gamma) \otimes X$, such that $v|_F = u|_F$, $\rho_u \subset \rho_v$, and*

$$(v - u)(T) \subset W, \qquad \operatorname{supp} v \subset u^{-1}(X \setminus \{0\}), \tag{20}$$

$$v(T) \subset \mathrm{co}(u(T)). \tag{21}$$

Proof. If $0 \in u(T)$, the conclusion follows easily by applying Theorem 16. If $0 \notin u(T)$, choose $x \in u(T)$. Applying Theorem 16 for $u_x := u - x \in C(T, X)$ shows that there exists $w \in C(T, \Gamma) \otimes X$, such that $\rho_{u_x} \subset \rho_w$, and

$$w|_F = u|_F - x, \quad x + (w - u)(T) \subset W, \quad w(T) \subset \mathrm{co}(u(T)) - x.$$

It is easy to check that $v := w + x \in C(T, \Gamma) \otimes X$ satisfies all required properties, since $\rho_u = \rho_{u_x} \subset \rho_w = \rho_v$ and $\operatorname{supp} v \subset T = u^{-1}(X \setminus \{0\})$. □

4.4. A final discussion

We shall now focus our attention on the connection between our results from Section 4.3, and a very general theorem on simultaneous approximation and interpolation in topological vector spaces, from Deutsch [4] and Singer [11]. We shall see that the factorization from Section 2.1 is a surprising key ingredient in establishing this connection.

Remark 18. For $X = \Gamma^n$, replacing [2] (17) and (21) by

$$v(T) \subset \mathrm{Span}(u(T)),$$

leads to weakened results which are consequences of the theorem from [4,11].

Indeed, assume the hypothesis of Theorem 16 holds, with given $\varepsilon > 0$ instead of the neighborhood W. Let us consider the vector spaces $E \subset Z \subset C_0(T, \Gamma^n)$,

$$Z := \{v \mid v(T) \subset \mathrm{Span}(u(T)),\ v^{-1}(\Gamma^n \setminus \{0\}) \subset u^{-1}(\Gamma^n \setminus \{0\}),\ \rho_u \subset \rho_v\},$$
$$E := C_{\mathrm{c}}(T, \Gamma^n) \cap \{v \in Z \mid \mathrm{supp}\, v \subset u^{-1}(\Gamma^n \setminus \{0\})\},$$

equipped with the supremum norm $\|v\|_\infty = \sup_{t\in T} \|v(t)\|$, the functionals $L_{t,j} \in Z^*$ $(t \in F,\ 1 \leqslant j \leqslant n)$ defined by $L_{t,j}v := v_j(t)$ (the jth component of $v(t) \in \Gamma^n$), and the neighborhood $V := \{v \in Z \mid \|v - u\|_\infty < \varepsilon\}$ of u in Z. If E is dense in Z, then applying the cited theorem from [4,11] shows the existence of a function $v \in E$, such that $\|v - u\|_\infty < \varepsilon$ and $v|_F = u|_F$.

It remains to prove that E is dense in Z. As $u \in Z$, we have $\rho_Z = \rho_u \subset \rho_E$. We see that (2) holds for the self-adjoint set

$$S := \{\varphi \in C_{\mathrm{b}}(T, \Gamma) \mid \rho_u \subset \rho_\varphi\}.$$

As S contains all components of functions from Z, we also have $\rho_S = \rho_u$, and so (3) holds. By Corollary 8, we deduce that $\overline{E}^{\mathrm{u}}_{C_0(T,\Gamma^n)} = \widetilde{E} \cap C_0(T, \Gamma^n)$, and consequently that $\overline{E}^{\mathrm{u}}_Z = \widetilde{E} \cap Z$. We are thus reduced to prove that $Z \subset \widetilde{E}$, which is equivalent to

$$\rho_E \subset \rho_u, \qquad Z(t) \subset \overline{E(t)} \text{ for every } t \in T.$$

We see now exactly where the problem is. We should find "sufficiently many" functions from E, and so we would need to apply Urysohn's lemma. We cannot do this directly, since T is arbitrary. As we shall see, the solution is to factorize the topological space T, as well as the functions spaces Z and E.

We can first simplify the original density problem by two successive reductions.

1. We can assume the subspace $\mathrm{Span}(u(T))$ is Γ^n, for if not, we may replace Γ^n by this subspace containing all ranges of functions from Z.
2. We can assume that $T = u^{-1}(\Gamma^n \setminus \{0\})$, since all functions from Z vanish on the complement of the latter open set.

Under the above assumptions we have

$$Z = \{v \in C_0(T, \Gamma^n) \mid \rho_u \subset \rho_v\}, \qquad E = C_{\mathrm{c}}(T, \Gamma^n) \cap Z.$$

Now the task seems to be simpler, but the problem is the same: T still is arbitrary. Let us consider the Hausdorff quotient space $T_u := T/\rho_u$. Since $\rho_u = \rho_Z \subset \rho_E$, all functions from $Z \supset E$ factorize as described in Section 2.1, thus leading to the functions spaces

$$\widehat{Z} = \{\widehat{v} \mid v \in Z\} = C_0(T_u, \Gamma^n), \quad \widehat{E} = C_{\mathrm{c}}(T_u, \Gamma^n) = C_{\mathrm{c}}(T_u, \Gamma) \otimes \Gamma^n. \tag{22}$$

[2] These restrictions do not fit well into the theorem from [4,11].

The density of E in Z is equivalent to that of $\widehat{E}$ in $\widehat{Z}$. As $\widehat{u} \in C_0(T_u, \Gamma^n)$, we deduce that $T_u = \widehat{u}^{-1}(\Gamma^n \setminus \{0\})$ is locally compact. Now by (22) it is clear that $\widehat{E}$ is dense in $\widehat{Z}$ (this being a known Stone–Weierstrass-type result).

From the above discussion we may conclude that factorization is by itself an interesting and efficient tool. It fits very well with Stone–Weierstrass results by establishing an optimal agreement between sets of vector-valued functions and sets of multipliers. Also, factorization converts sets of multipliers into separating sets of multipliers defined on Hausdorff spaces. Stone–Weierstrass theorems obtained via factorization are general and powerful.

For other recent Stone–Weierstrass-type results for continuous functions, but in a different approach, we refer the reader to Bustamante and Montalvo [3], and Briem [1].

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